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TENSOR PRODUCTS OF SOME IRREDUCIBLE UNITARY REPRESENTATIONS OF THE UNIVERSAL COVERING GROUP OF SU(1,1) THAT DEFINE REPRESENTATIONS OF SU(1,1)

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Abstract. We consider tensor products of irreducible unitary representations of the universal covering group G of $\mathrm{SU}(1,1)$ belonging to the so-called positive discrete series of representations of G in the terminology used by Sally in his well-known book [P. J. Sally, Analytic Continuation of the Irreducible Unitary Representations of the Universal Covering Group of $\mathrm{SL}(2,\mathbb{R})$, Memoirs Amer. Math. Soc. No. 69, American Mathematical Society, Providence, RI, 1967]. We show that, for the special values of the parameters that are indicated in the paper, these products take the kernel of the natural homomorphism of G onto $\mathrm{SU}(1,1)$ to the identity operator, and hence define representations of the group $\mathrm{SU}(1,1)$. We then study the decomposition of this representation into the direct sum of irreducible representations and study some tensor products of unitary pseudorepresentations of $\mathrm{SU}(1,1)$.

§ 1. Introduction

As is well known [1], an irreducible unitary representation of Hermitian-symmetric simple Lie groups can be deformed continuously in the class of pure pseudorepresentations. The general natural problem of whether or not a unitary pure pseudorepresentation admits a decomposition into a direct

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integral of pseudorepresentations is open, which makes the problem of decomposing the tensor products of irreducible unitary pseudorepresentations more or less artificial (recall that the tensor product of two bounded pseudorepresentations is well defined [2–4]. In this note, we restrict ourselves to pseudorepresentations of $\mathrm{SU}(1,1)$, or $\mathrm{SL}(2,\mathbb{R})$, which, as is well known, are (tensor) products of irreducible unitary representations of the universal covering group of $\mathrm{SU}(1,1)$ by an appropriate power of the Guichardet–Wigner pseudocharacter on G [3]. For two pseudorepresentations for which these powers are mutually inverse, the tensor product takes the value equal to the identity operator on the whole kernel of the natural homomorphism taking G onto $\mathrm{SU}(1,1)$, and this means that the tensor product thus obtained defines an ordinary representation of $\mathrm{SU}(1,1)$, whose decomposition into irreducible unitary representations can be studied by the standard tools.

§ 2. Preliminaries

We use the terminology and notation of [5]. It is useful for our purposes that the group $\mathrm{SL}(2,\mathbb{R})$ is considered in the form $\mathrm{SU}(1,1)$.

Recall one of the standard parametrizations of G. Following [5], we write out the elements of G by pairs,

$$G = ((\gamma,\omega) \mid \gamma \in \mathbb{C}, |\gamma| < 1, \omega \in \mathbb{R}),$$

where

$$(\gamma,\omega) = \{\omega, a_0\} = \left\{\omega, \begin{pmatrix} e^{i\omega}(1-|\gamma|^2)^{-1/2} & e^{-i\omega}\overline{\gamma}(1-|\gamma|^2)^{-1/2} \\ e^{i\omega}\gamma(1-|\gamma|^2)^{-1/2} & e^{-i\omega}(1-|\gamma|^2)^{-1/2} \end{pmatrix}\right..$$

Let D be the open unit disk in the complex plane. Let $H_{2,h}(D)$, $h > 1/2$, be the Hilbert space defined by the inner product

$$(f,g)_h = \frac{2h-1}{\pi}\int_D f(z)\overline{g(z)}(1-|z|^2)^{2h-2}\,dxdy, \qquad z = x+iy;$$

the inner product in the space $H_{2,1/2}(D)$ is defined as the limit of $(f,g)_h$ as $h \to 1/2+$; for the definition of the space $H_{2,h}(D)$, $1/2 > h > 0$, see [5]. By $[U^+_{a,h}$ we mean the representation given by the formulas

$$[U^+_{a,h}f](z) = e^{2i\omega h}(1-|\gamma|^2)^h(1+\overline{\gamma}z)^{-2h}f(z\cdot a_0), \qquad a = (\omega, a_0), \quad h > 0,$$

$$a_0 = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, \qquad z\cdot a_0 = \frac{z\alpha+\beta}{z\overline{\beta}+\overline{\alpha}}, \quad z \in D, \quad f \in H_{2,h}(D).$$

As is well known, the restriction of the representation $U(\cdot,h)+$ to the subgroup

$$K_0 = \left\{\omega, \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix}\right\}$$

is the direct sum of the one-dimensional unitary characters of K_0 of the form $\chi_{h+2m} \colon \omega \mapsto e^{2\pi i(h+2m)\omega}$, $m \geq 0$.

This observation immediately implies the following assertion.

Theorem 1. *The restriction to K_0 of the tensor product of two representations of positive discrete series $U^+_{h_1}$ and $U^+_{h_2}$ is formed by the direct sum of the characters $\chi_{h_1+h_2+2m_1+2m_2}$, where m_1 and m_2 range over all nonnegative integers.*

The mapping $F\colon G \to R$ defined by the rule $a = (\omega, a) \mapsto \omega$, $a \in G$, is a quasicharacter on G, the so-called Guichardet–Wigner quasicharacter. Let φ be the corresponding pseudocharacter on G, which is called the Guichardet–Wigner pseudocharacter [2, 3].

Let s be the signed distance from h to the nearest half-integer ($h+s$ is a nearest half-integer to h and $|s| \leq 1/4$). Let $U^+_{\cdot,h,s}$ be the deformation of $U^+_{\cdot,h}$ defined by the formula

$$[U^+_{a,h,s}f](z) = e^{2is\varphi(a)}e^{2i\omega h}(1-|\gamma|^2)^h(1+\overline{\gamma}z)^{-2h}f(z\cdot a_0), \; a = (\omega, a_0), \; h > 0,$$

$$a_0 = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, \qquad z\cdot a_0 = \frac{z\alpha+\beta}{z\overline{\beta}+\overline{\alpha}}, \quad z \in D, \quad f \in H_{2,h}(D).$$

It is clear that $U^+_{a,h,s} = 1_{H_{2,h}(D)}$, and hence $U^+_{\cdot,h,s}$ can be regarded as a pseudorepresentation of $\mathrm{SU}(1,1)$.

§ 2. Main result

Consider the case in which $h_1 + h_2$ is an integer.

Theorem 2. *Let ρ be the tensor product of the representations $U(\cdot,h_1)^+$ and $U(\cdot,h_2)^+$, $h_1, h_2 > 0$, where $h_1 + h_2 = k/2$ for an integer k. Then ρ can be regarded as a representation of $\mathrm{SU}(1,1)$ which is the orthogonal direct sum of the representations equivalent to $U(\cdot,r)$, where r ranges over the set $\{h_1+h_2+m, m \in \mathbb{Z}\}$.*

Proof. The above list of characters of the restriction to K_0 of the representation $U(\cdot, h)+$ and the corresponding lists of restrictions to H_0 of the representations of the other series show that the tensor product in question has a direct summand of the form $U(\cdot, h_1 + h_2)$. In the orthogonal complement of the subspace of this representation, the only possibility to find a summand of the form $U(\cdot, h)+$ in this complement is a direct summand of the form $U(\cdot, h_1 + h_2 + 1)$. Continuing this process by induction, we obtain the assertion of the theorem.

§ 4. Discussion

Recall that the direct integral decomposition of a unitary pseudorepresentation is an open question. Nevertheless, for the unitary pseudorepresentations of $\mathrm{SU}(1,1)$ constructed above, namely, for the tensor products of deformations of discrete series representations, the problem on the orthogonal decomposition into a direct sum of unitary pseudorepresentations can be solved in the affirmative. For the generalities concerning tensor products of quasirepresentations and pseudorepresentations, see [2].

Theorem 3. *Let ρ be the tensor product of pseudorepresentations of the form $U(\cdot, h_1, s_1)^+$ and $U(\cdot, h_2, s_2)^+$, $h_1, h_2 > 0$. Then ρ can be regarded as a pseudorepresentation of* $\mathrm{SU}(1,1)$ *which is the orthogonal direct sum of the pseudorepresentations equivalent to $U(\cdot, r, s_1 + s_2)$, where r ranges over the set $\{h_1 + h_2 + m, m \in \mathbb{Z}\}$.*

Proof. The pseudorepresentation in question can be viewed as a product of the tensor product of two ordinary representations of the universal covering group G of $\mathrm{SU}(1,1)$ by a numerical function which is a product of two exponentials of the Guichardet–Wigner pseudocharacters on G. The direct sum decomposition of the tensor product by itself can be carried out just as the decomposition in Theorem 2; the subsequent multiplication by the numerical function $e^{2i(s_1+s_2)\varphi(a)}$, $a \in G$, establishes the final list of one-dimensional characters entering the restriction of the tensor product in question to K_0 which leads to the final list of quasirepresentations entering the "corrected" direct sum of the pseudorepresentations of $\mathrm{SU}(1,1)$ involved in the decomposition.

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